

ON GENERATORS AND REPRESENTATIONS OF THE SPORADIC SIMPLE GROUPS

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ABSTRACT. In this paper we determine the irreducible projective representations of sporadic simple groups over an arbitrary algebraically closed field F , whose image contains an almost cyclic matrix of prime-power order. A matrix M is called cyclic if its characteristic and minimum polynomials coincide, and we call M almost cyclic if, for a suitable $\alpha \in F$, M is similar to $\text{diag}(\alpha \cdot \text{Id}_h, M_1)$, where M_1 is cyclic and $0 \leq h \leq n$. The paper also contains results on the generation of sporadic simple groups by minimal sets of conjugate elements.

1. INTRODUCTION

Problems on group generation by suitable subsets have been the subject of intensive research since the origins of group theory. Apart from its intrinsic interest, this subject gives rise to many applications, and has been used extensively in answering questions on many topics within group theory. In particular, it is well known that some aspects of representations of finite groups are connected to the existence of generating sets of a certain kind.

In this paper we are interested in the generation of a group by sets of conjugates of a given group element. Concerning the sporadic simple groups, one of the first results related to generation by conjugates is due to I. Zisser ([23]), who determined the 'covering number' of each sporadic group (see also Table 1 on p. 554 in [11]). This is the minimum number $m = m(G)$ such that for every non-trivial conjugacy class C of G one has $G = \bigcup_{i=1}^m C^i$, where $C^i = \{g \in G : g = x_1 x_2 \cdots x_i \text{ for some } x_j \in C\}$. If m is the covering number of a simple group G , then G can be generated by $m + 1$ suitable elements from any given non-trivial class C (see [11], Lemma 2.12). It was shown in [23] that $m(G) \leq 4$ unless $G = Fi_{22}, Fi_{23}$, in which case $m(G) = 6$.

In [11] (see Lemma 7.6 and Table 1, p. 554), these bounds were slightly improved for some of the groups, combining Zisser's results with the information provided by the Atlas of finite simple groups ([1], [2]) and the knowledge of lower bounds for the degrees of representations of sporadic groups. However, no specific attention to the order of the elements of a given class C was paid there, and so it remained open whether and when G might be generated by, say, two elements of C . More recently, the problem of the generation of a simple group

2010 *Mathematics Subject Classification.* 20F05, 20C15, 20C20, 20C34, 20C40.

Key words and phrases. Sporadic simple groups, Generation by conjugates, Irreducible representations, Eigenvalue multiplicities.

The second author was supported by FEMAT.

by sets of conjugate involutions satisfying certain specific conditions has been considered by J. Ward in his PhD dissertation (see [18]).

In this paper we determine, for every finite sporadic simple group G and most conjugacy classes C of G , the minimum number $\alpha_G(g)$ of conjugates of $g \in C$ required to generate G . The results obtained are summarized in Theorem 3.1 below.

Theorem 3.1 can be viewed as a refinement and improvement of the results of [23] and [11]. Indeed, in comparison with the bounds given there, it provides better estimates for each sporadic group G , except when $G \in \{M_{11}, J_1, J_2, M\}$ (in the Atlas notation).

We also find out (see Section 3) that, unless $G = M$, whenever the order of $g \in G$ is greater than 4, then $\alpha_G(g) = 2$, and $\alpha_G(g)$ conjugates of g can be chosen, such that their product in a suitable ordering has order equal to the highest prime divisor of $|G|$.

We emphasize that Theorem 3.1 cannot be proven via straightforward computations, except when the groups are very small. Instead, our approach makes use of character theory and the character tables of the sporadic groups, and can be outlined as follows.

Given a finite group G , and $k \geq 3$ (not-necessarily distinct) conjugacy classes of G C_1, \dots, C_k , there exists a formula in terms of the values of irreducible characters of G at C_1, \dots, C_k (see Section 2), which gives the number $\Delta_G(C_1, \dots, C_k)$ of solutions of the equation $g_1 g_2 \cdots g_{k-1} = g_k$, where $g_i \in C_i$ ($1 \leq i \leq k-1$) and g_k is a fixed element of the class C_k . Next compute, for every maximal subgroup H of G that meets every class C_1, \dots, C_k , the number $\Delta_H(c_1, \dots, c_k)$ for all the H -conjugacy classes c_1, \dots, c_k such that $c_i \subseteq H \cap C_i$. Denote by $\Sigma_H(C_1, \dots, C_k)$ the sum of all such structure constants $\Delta_H(c_1, \dots, c_k)$. Suppose that

$$\Delta_G(C_1, \dots, C_k) > \sum h(g_k, H) \cdot \Sigma_H(C_1, \dots, C_k),$$

where $h(g_k, H)$ is the number of the distinct conjugates of H containing g_k , and the sum is taken over the representatives H of the G -classes of maximal subgroups of G containing elements of all the classes C_1, \dots, C_k . Then there exist elements $g_i \in C_i$ such that $G = \langle g_1, \dots, g_{k-1} \rangle$. In our situation, $C_1 = \dots = C_{k-1}$, and in most cases the class C_k plays a special role. Namely, most often we find that C_k can be chosen uniformly, that is C_k can be chosen to be the same class for any choice of C_1 . Recall that a group G is said to be (C_1, \dots, C_k) -generated if there exist $g_i \in C_i$ ($1 \leq i \leq k$) such that $g_1 \cdots g_{k-1} = g_k$ and $G = \langle g_1, \dots, g_{k-1} \rangle$. Thus, our computations yield results on (C_1, \dots, C_k) -generation for $C_1 = \dots = C_{k-1}$. (See Section 2 for further details.)

The results on generating sets, described in detail in Section 3, allow us to study in an efficient way certain properties of the eigenvalues of matrices in the representations of the sporadic groups. More precisely, we are committed to determine all the projective irreducible representations of sporadic groups for which there exist elements of prime-power order represented by so-called almost cyclic matrices.

The notion of almost cyclic matrix is a generalization of the notion of cyclic matrix. Namely, let V be a finite dimensional vector space over a field F . Cyclic matrices are exactly those whose characteristic polynomial coincides with the minimum one. (Note that a matrix $X \in \text{End } V$ is cyclic if and only if the $F\langle X \rangle$ -module V is cyclic, that is, is generated by a single element. This is a standard terminology of ring theory, and the source

of the term ‘cyclic matrix’. Matrices with simple spectrum often arising in applications are cyclic.)

Now, we define a matrix $M \in \text{Mat}(n, F)$ to be almost cyclic if there exists $\alpha \in F$ such that M is similar to $\text{diag}(\alpha \cdot \text{Id}_h, M_1)$, where M_1 is cyclic and $0 \leq h \leq n$.

Observe that, if \overline{F} denotes the algebraic closure of F , and λJ for $\lambda \in \overline{F}$ denotes a Jordan block with eigenvalue λ , then a matrix M_1 is cyclic if and only if M_1 has Jordan form $\text{diag}(\lambda_1 J_1, \dots, \lambda_s J_s)$, where the λ_j ’s, $1 \leq j \leq s$, are pairwise distinct. In particular, suppose that $M = \text{diag}(\alpha \cdot \text{Id}_h, M_1)$, with $0 \leq h < n$, is non-scalar of order p^a for a prime p , and set $\ell = \text{char } F$. Then M is almost cyclic if and only if the eigenvalues of M_1 are pairwise distinct when $\ell \neq p$, and if and only if M_1 consists of a single Jordan block when $\ell = p$.

Almost cyclic matrices arise naturally in the study of matrix groups over finite fields. Pseudo-reflections are important examples, as well as unipotent matrices with Jordan form consisting of a single non-trivial block.

A key contribution to the subject is a paper by Guralnick, Penttila, Praeger and Saxl ([10]), in which the authors classified linear groups over finite fields generated by ‘Dempwolff elements’. Let $V = V(n, q)$ be an n -dimensional vector space over a finite field of order q , $H = GL(V) = GL(n, q)$ and $g \in H$. We say that g is a Dempwolff element if $|g| = p$ for some prime p with $(p, q) = 1$ and g acts irreducibly on $V^g := (\text{Id} - g)V$. U. Dempwolff in [5] initiated the study of subgroups of $GL(n, q)$ generated by such elements, obtaining a number of valuable results. The main restriction in [5] is the assumption that $2 \dim V^g > \dim V$, and this assumption is held in [10]. Clearly, Dempwolff elements are almost cyclic (and are reflections if $p = 2$).

Possibly, the strongest motivation to study groups containing an almost cyclic matrix is to contribute to the recognition of linear groups and finite group representations by a property of a single matrix. Answers to problems of this kind are often required in several applications.

In fact, there is an extensive literature containing important results related more or less strictly to our subject, both before and after Dempwolff’s work (e.g. results due to Hering, Wagner, Suprunenko, Huffman, Wales, Tiep, Guralnick, Saxl and others). For a more detailed description of this literature, see, e.g., [15], [21] and [6].

The present paper may be viewed as a necessary piece of a project initiated in [6] and [7]. The paper [6] classifies the irreducible cross-characteristic representations of finite quasi-simple groups of Lie type, for which there exist unipotent elements represented by almost cyclic matrices. The paper [7] analyzes the occurrence of almost cyclic semisimple elements of prime-power order in cross-characteristic representations of finite quasi-simple groups of Lie type.

Our goal here is to examine the irreducible representations of the finite simple sporadic groups and their covering groups. The techniques exploited are of computational nature, and thus differ substantially from those of [6] and [7]. As it should be expected, substantial use is made of the mass of information available in the Atlas and Modular Atlas of finite groups ([1], [2]), together with the routines existing or implementable in GAP and MAGMA ([9], [3]). The results we have obtained are collected in Section 7.

Finally, we note that the connection between the two problems we address in the paper (generation by conjugates, existence of elements representable by almost cyclic matrices) is based on Lemma 2.1 below, which bounds from above the degree of a linear group G generated by almost cyclic matrices conjugate to a given $g \in G$, in terms of the order of g and $\alpha_G(g)$. In fact, this was our original motivation for studying the above generation problem in detail. Furthermore, Lemma 2.1 together with other machinery described in Section 2 are essential in order to reduce significantly the amount of computations necessary to obtain the results stated in Section 7.

Notation. Throughout the paper we assume F to be an algebraically closed field of characteristic ℓ .

For an $(n \times n)$ -matrix A over a field F , we denote by $m_A(x)$ and $p_A(x)$ the minimum and the characteristic polynomial of A , respectively.

Following the conventions introduced in the Atlas ([1]), we denote by nX a conjugacy class of a group G consisting of elements of order n . We warn that the letter X is chosen according to the labelling adopted in GAP ([9]). The notation used for the 26 simple sporadic groups is the standard one. The known maximal subgroups of each sporadic group can be found in [19]. In the text and tables of the present paper, a representation of a given sporadic group will usually be indicated only by its degree. We emphasise here that this shortcut is justified by the fact that, when the group has two or more representations of the same degree, the results we will obtain turn out to be independent of the choice of the representation.

2. BASIC MACHINERY

The following elementary result establishes a useful connection between the occurrence of almost cyclic matrices in representations of irreducible linear groups and their generation by conjugates.

Lemma 2.1. *Let F be an algebraically closed field. If $G < GL(n, F)$ is a finite irreducible linear group generated by m almost cyclic elements g_i of the same order d (modulo $Z(G)$), then*

$$n \leq m(d - 1).$$

Proof. Let $V = V(n, F)$ be the underlying vector space of $GL(n, F)$. Let α_i be an eigenvalue of g_i with eigenspace of maximal dimension. Define $V_i = \text{Im}(g_i - \alpha_i I)$. Clearly, V_i is $\langle g_i \rangle$ -invariant. Moreover, considering the action of g_i induced on the quotient space V/V_i , we observe that $g_i(V_i + x) = V_i + \alpha_i x$. Thus, for each $i = 1, \dots, m$,

$$g_i\left(\sum_{j=1}^m V_j\right) = g_i(V_i + \sum_{j \neq i} V_j) \subseteq V_i + \alpha_i \sum_{j \neq i} V_j = \sum_{j=1}^m V_j.$$

This means that $\sum_{j=1}^m V_j$ is G -invariant, whence, as G is irreducible, $V = \sum_{j=1}^m V_j$. On the other hand, we claim that, for each i , $\dim V_i \leq d - 1$, whence $n \leq m(d - 1)$, as required. Let $g_i^d = \lambda I_n$, for some $\lambda \in F$. First, suppose that g_i is cyclic. Then $m_{g_i}(x) = p_{g_i}(x)$ divides $x^d - \lambda$, which implies $n \leq d$. It follows that $\dim V_i = n - \dim \text{Ker}(g_i - \alpha_i I) = n - 1 \leq d - 1$. Next, suppose that g_i is almost cyclic, but not cyclic. This means that g_i is similar to a

matrix of shape $\text{diag}(\alpha_i, \dots, \alpha_i, \bar{g})$, where \bar{g} is a cyclic matrix of size $k > 0$. We have two possibilities. First, α_i is not an eigenvalue of \bar{g} . Then $m_{g_i}(x) = (x - \alpha_i)m_{\bar{g}}(x)$ divides $x^d - \lambda$, whence $1 + k \leq d$ and $\dim V_i = n - (n - k) = k \leq d - 1$. Next, suppose that α_i is an eigenvalue of \bar{g} . Then \bar{g} is similar to a matrix of shape $\text{diag}(J, \tilde{g})$, where J is the (unique) Jordan block corresponding to the eigenvalue α_i and \tilde{g} is cyclic. Let t be the size of J . Then $m_{g_i}(x) = (x - \alpha_i)^t m_{\tilde{g}}(x)$ divides $x^d - \lambda$. This implies $k \leq d$ and therefore again $\dim V_i = n - (n - k + 1) = k - 1 \leq d - 1$. So the statement is proven. \square

An immediate consequence of the previous Lemma is that, if $\Phi : G \rightarrow GL(n, F)$ is an irreducible faithful representation of a finite group G , which can be generated by m conjugate elements g_i of order d such that $\Phi(g_i)$ is almost cyclic, then $\dim \Phi = n \leq m(d-1)$. Therefore, for a fixed d , the smaller m is, the smaller will be the degree and hence the number of the representations to be examined when searching for elements of order d of G represented by almost cyclic matrices. For example, it will turn out that, for every simple sporadic group G different from M and for every conjugacy class C of elements of G of prime-power order $d > 4$, two suitable elements of C are enough to generate G . That is, we can choose $m = 2$ in the bound given above. This drastically reduces the computations necessary to prove the results stated in Section 7.

In view of the above considerations, we need to exploit results on the generation of a group G by conjugates. Furthermore, whenever some necessary data on maximal subgroups are missing in GAP (as in the case of some covering groups), or the maximal subgroups of the group G are not completely known (as in the case of the Monster), it will also be useful to know the order of the product of certain pairs of conjugate elements. This makes all the more convenient a systematic use of the ‘structure constants method’ (as applied, e.g., in [8]), though clearly the information obtained in this way is generally more precise than strictly required for our purposes in most cases. So, we now recall, for the reader’s sake, the basics of the ‘structure constants method’.

Given a finite group G , let C_1, \dots, C_k be $k \geq 3$ (not-necessarily distinct) conjugacy classes of G . Denote by $\Delta_G = \Delta_G(C_1, \dots, C_k)$ the number of distinct k -tuples (g_1, \dots, g_k) , where $g_i \in C_i$ ($1 \leq i \leq k-1$), g_k is a fixed element of the class C_k , and $g_1 g_2 \cdots g_{k-1} = g_k$. This structure constant can be computed using the (complex) character table. Namely, it is given by the formula

$$\Delta_G(C_1, \dots, C_k) = \frac{|C_1| \cdots |C_{k-1}|}{|G|} \cdot \sum_{i=1}^r \frac{\chi_i(g_1) \chi_i(g_2) \cdots \chi_i(g_{k-1}) \overline{\chi_i(g_k)}}{(\chi_i(1))^{k-2}},$$

where χ_1, \dots, χ_r are the irreducible complex characters of G .

Next, for a fixed $g_k \in C_k$ denote by $\Delta_G^*(C_1, \dots, C_k)$ the number of distinct k -tuples (g_1, \dots, g_k) such that $g_i \in C_i$ ($1 \leq i \leq k-1$), $g_1 \cdots g_{k-1} = g_k$, and $G = \langle g_1, \dots, g_{k-1} \rangle$. If $\Delta_G^*(C_1, \dots, C_k) > 0$, the group G is said to be (C_1, \dots, C_k) -generated. For our purposes, we aim to find the minimal k for which $\Delta_G^*(C_1, \dots, C_k)$ is positive for certain classes C_1, \dots, C_k of elements of a given order.

To this end, let H be a maximal subgroup of G containing a fixed element $g_k \in C_k$, and denote by $\Sigma_H(C_1, \dots, C_k)$ the number of distinct $(k-1)$ -tuples $(g_1, \dots, g_{k-1}) \in C_1 \times \cdots \times C_{k-1}$ such that $g_1 \cdots g_{k-1} = g_k$ and $\langle g_1, \dots, g_{k-1} \rangle \leq H$. The value of $\Sigma_H(C_1, \dots, C_k)$ can be

obtained as the sum of the structure constants $\Delta_H(c_1, \dots, c_k)$ of H for all the H -conjugacy classes c_1, \dots, c_k such that $c_i \subseteq H \cap C_i$.

Now, the following holds:

Lemma 2.2 (e.g. see [8]). *Let G be a finite group and let H a subgroup of G containing a fixed element x . Denote by $h(x, H)$ the number of the distinct conjugates of H containing x . If $(|x|, |N_G(H) : H|) = 1$, then*

$$h(x, H) = \sum_{i=1}^s \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where x_1, \dots, x_s are representatives of the $N_G(H)$ -conjugacy classes fused to the G -class of x .

As a consequence, we obtain an useful lower bound for $\Delta_G^*(C_1, \dots, C_k)$. Namely:

$$\Delta_G^*(C_1, \dots, C_k) \geq \Theta_G(C_1, \dots, C_k),$$

where

$$\Theta_G(C_1, \dots, C_k) = \Delta_G(C_1, \dots, C_k) - \sum h(g_k, H) \Sigma_H(C_1, \dots, C_k),$$

g_k is a representative of the class C_k , and the sum is taken over the representatives H of the G -classes of maximal subgroups of G containing elements of all the classes C_1, \dots, C_k .

Unless $G = M$, $\Theta_G = \Theta_G(C_1, \dots, C_k)$ can be computed using the GAP routines. Thus, if $\Theta_G > 0$, certainly G is (C_1, \dots, C_k) -generated. In particular, in the case when $C_1 = \dots = C_{k-1} = C$, this tells us that G can be generated by $k - 1$ elements suitably chosen from the class C .

Furthermore, if $\Theta \leq 0$, in some cases one can prove that the group G is not (C_1, \dots, C_k) -generated (in particular, that $k - 1$ is actually the minimum number of elements from a given class C necessary to generate G), with the help of the following Lemma (of which we give the straightforward proof for the sake of clarity):

Lemma 2.3. (cf. [4]) *Let G be a finite centerless group. If $\Delta_G^*(C_1, \dots, C_k) > 0$, then $\Delta_G^*(C_1, \dots, C_k) \geq |C_G(g_k)|$, for any $g_k \in C_k$.*

Proof. As $\Delta_G^*(C_1, \dots, C_k) > 0$, for any fixed element $g_k \in C_k$ there exists at least one $(k - 1)$ -tuple (g_1, \dots, g_{k-1}) such that

$$(*) \quad g_i \in C_i, \quad g_1 \cdots g_{k-1} = g_k \text{ and } G = \langle g_1, \dots, g_{k-1} \rangle.$$

Let $x \in C_G(g_k)$. Then

$$x(g_1 \cdots g_{k-1})x^{-1} = (xg_1x^{-1}) \cdots (xg_{k-1}x^{-1}) = xg_kx^{-1} = g_k.$$

Thus, the $(k - 1)$ -tuple $(xg_1x^{-1}, \dots, xg_{k-1}x^{-1})$ also satisfies $(*)$. Furthermore, if x_1, x_2 are distinct elements of $C_G(g_k)$, then the $(k - 1)$ -tuples $(x_1g_1x_1^{-1}, \dots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, \dots, x_2g_{k-1}x_2^{-1})$ are also distinct, since $Z(G) = \{1\}$. This implies that there are at least $|C_G(g_k)|$ $(k - 1)$ -tuples (g_1, \dots, g_{k-1}) satisfying $(*)$. That is, $\Delta_G^*(C_1, \dots, C_k) \geq |C_G(g_k)|$. \square

So, obviously, if $\Delta_G(C_1, \dots, C_k) < |C_G(g_k)|$ for $g_k \in C_k$, the previous Lemma tells us that G cannot be (C_1, \dots, C_k) -generated.

Recall that a non-scalar $g \in GL(V)$ is called pseudoreflection if g acts scalarly on a hyperplane of V . Observe that the matrix of a pseudoreflection is almost cyclic. We will use the following result:

Lemma 2.4. *Let $G < GL(n, F)$ be a finite irreducible linear group generated by pseudoreflections. Then G cannot be a sporadic simple group.*

Proof. The statement follows immediately from the classification theorems due to A.E. Zalesski and V.N. Serezhkin (see [22]) and A.O. Wagner (see [16, 17]). \square

Finally, for the reader's convenience, we quote the following:

Proposition 2.5 ([20]). *Let G be a quasi-simple finite sporadic group and let $Z(G)$ be its center. Let ℓ be a prime and P be a Sylow ℓ -subgroup of G . Let K be a splitting field for G of characteristic ℓ and let M be a faithful irreducible KG -module. Suppose that P is cyclic and $M|_P$ contains no submodule isomorphic to the regular KP -module. Then $\ell > 2$ and one of the following holds:*

- (1) $G = M_{11}$, $|P| = 11$ and $\dim M = 9$ or 10 ;
- (2) $G = M_{23}$, $|P| = 23$ and $\dim M = 21$;
- (3) $G/Z(G) = M_{12}$ or M_{22} , $|P| = 11$, $\dim M = 10$ and $|Z(G)| = 2$;
- (4) $G/Z(G) = Suz$, $|P| = 7$ or 13 , $\dim M = 12$ and $|Z(G)| = 6$;
- (5) $G/Z(G) = J_3$, $|P| = 19$, $\dim M = 18$ and $|Z(G)| = 3$;
- (6) $G/Z(G) = Ru$, $|P| = 29$, $\dim M = 28$ and $|Z(G)| = 2$;
- (7) $G/Z(G) = J_2$, $|P| = 7$, $\dim M = 6$ and $|Z(G)| = 2$;
- (8) $G/Z(G) = Co_1$, $|P| = 13$ and $\dim M = 24$;
- (9) $G = J_1$, $|P| = 11$ and $\dim M = 7$.

Conversely, in all these cases $M|_P$ does not contain the regular KP -submodule. So $\dim M < |P|$, except for the following cases, where $|P| = \ell$ and $\dim M = 2(\ell - 1)$:

- (i) $\ell = 7$, $G/Z(G) = Suz$;
- (ii) $\ell = 13$, $G/Z(G) = Co_1$.

Finally, if g is a generator of P , the degree of the minimum polynomial of $g|_M$ equals $\dim M$ (and hence $g|_M$ is cyclic), except for the cases (i) and (ii), where the degree is $(\dim M)/2$.

An immediate consequence of the above Proposition is the following, which will be useful in the sequel:

Corollary 2.6. *Let G be a quasi-simple finite sporadic group and let F be an algebraically closed field of positive characteristic ℓ . Let Φ be a faithful irreducible representation of G over F . Suppose that $P = \langle g \rangle$ is a Sylow ℓ -subgroup of G of order ℓ . The following holds:*

- (1) if $\dim \Phi = \ell$, then $\Phi(g)$ is cyclic;
- (2) if $\dim \Phi = \ell + 1$, then $\Phi(g)$ is almost cyclic.

Proof. If $G/Z(G) \in \{Suz, Co_1\}$, then $\dim \Phi > \ell + 1$ (e.g. see [13, 12]). By Proposition 2.5, it follows that $\Phi|_P$ contains as a constituent the regular representation of P . The statement follows by degree reasons. \square

3. THE SPORADIC GROUPS: GENERATION BY CONJUGATES

In this section we describe the results that we have obtained on the generation by sets of conjugate elements of a sporadic simple group G , exploiting the machinery introduced in Section 2. We have made a systematic use of GAP in order to analyze the generation of G by conjugates via the ‘structure constants method’. It turns out that this gives, for each element g of order $n > 1$, a set of conjugates of g generating G , which is in most cases of minimal size, though not always. In the latter case, a generating set of minimal size has been obtained by direct computation, using either GAP or MAGMA ([3]), except when the group involved is too large. More precisely, the results obtained are the following:

3.1. Let nX denote a conjugacy class of G consisting of elements of order $n > 1$. Computing via the GAP routines Θ_G , and thus obtaining a lower bound for Δ_G^* , we get the following:

- $G = M_{11}$. Then G is $(nX, nX, 11a)$ -generated for $nX \neq 2a, 3a$, while it is $(3a, 3a, 8a)$ -generated and $(2a, 2a, 2a, 11a)$ -generated;
- $G = M_{12}$. Then G is $(nX, nX, 11a)$ -generated for $nX \neq 2a, 2b, 3a$, while it is $(3a, 3a, 6a)$ -generated, $(2a, 2a, 2a, 11a)$ -generated and $(2b, 2b, 2b, 6a)$ -generated;
- $G = J_1$. Then G is $(nX, nX, 19a)$ -generated for $nX \neq 2a$, and $(2a, 2a, 2a, 19a)$ -generated;
- $G = M_{22}$. Then G is $(nX, nX, 11a)$ -generated for $nX \neq 2a$, and $(2a, 2a, 2a, 11a)$ -generated;
- $G = J_2$. Then G is $(nX, nX, 7a)$ -generated for $nX \neq 2a, 2b, 3a, 4a$, while it is $(4a, 4a, 5c)$ -generated, $(nX, nX, nX, 7a)$ -generated for $nX = 2b, 3a$, and $(2a, 2a, 2a, 2a, 7a)$ -generated;
- $G = M_{23}$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a$, and $(2a, 2a, 2a, 23a)$ -generated;
- $G = HS$. Then G is $(nX, nX, 11a)$ -generated for $nX \neq 2a, 2b, 4a$, and $(nX, nX, nX, 11a)$ -generated for $nX = 2a, 2b, 4a$;
- $G = J_3$. Then G is $(nX, nX, 19a)$ -generated for $nX \neq 2a$, and $(2a, 2a, 2a, 19a)$ -generated;
- $G = M_{24}$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a, 2b$, and $(nX, nX, nX, 23a)$ -generated for $nX = 2a, 2b$;
- $G = McL$. Then G is $(nX, nX, 11a)$ -generated for $nX \neq 2a, 3a$, and $(nX, nX, nX, 11a)$ for $nX = 2a, 3a$;
- $G = He$. Then G is $(nX, nX, 17a)$ -generated for $nX \neq 2a, 2b, 3a$, while it is $(3a, 3a, 8a)$ -generated and $(nX, nX, nX, 17a)$ for $nX = 2a, 2b$;
- $G = Ru$. Then G is $(nX, nX, 29a)$ -generated for $nX \neq 2a, 2b$, and $(nX, nX, nX, 29a)$ -generated for $nX = 2a, 2b$;
- $G = Suz$. Then G is $(nX, nX, 13a)$ -generated for $nX \neq 2a, 2b, 3a$, $(nX, nX, nX, 13a)$ -generated for $nX = 2a, 2b$, and $(3a, 3a, 3a, 3a, 13a)$ -generated;
- $G = O'N$. Then G is $(nX, nX, 31a)$ -generated for $nX \neq 2a$, and $(2a, 2a, 2a, 31a)$ -generated;

- $G = Co_3$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a, 2b, 3a$, while it is $(3a, 3a, 15a)$ -generated and $(nX, nX, nX, 23a)$ -generated for $nX = 2a, 2b$;
- $G = Co_2$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a, 2b, 2c, 4a$, while it is $(4a, 4a, 10a)$ -generated, $(nX, nX, nX, 23a)$ -generated for $nX = 2b, 2c$, and $(2a, 2a, 2a, 2a, 23a)$ -generated;
- $G = Fi_{22}$. Then G is $(nX, nX, 13a)$ -generated for $nX \neq 2a, 2b, 2c, 3a, 3b$, while it is $(nX, nX, nX, 13a)$ -generated for $nX = 2b, 2c, 3a, 3b$ and $(2a, 2a, 2a, 2a, 2a, 13a)$ -generated;
- $G = HN$. Then G is $(nX, nX, 19a)$ -generated for $nX \neq 2a, 2b$, and $(nX, nX, nX, 19a)$ -generated for $nX \neq 2a, 2b$;
- $G = Ly$. Then G is $(nX, nX, 67a)$ -generated for $nX \neq 2a, 3a$, and $(nX, nX, nX, 67a)$ -generated for $nX = 2a, 3a$;
- $G = Th$. Then G is $(nX, nX, 31a)$ -generated for $nX \neq 2a$ and $(2a, 2a, 2a, 31a)$ -generated;
- $G = Fi_{23}$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a, 2b, 2c, 3a, 3b$, $(nX, nX, nX, 23a)$ -generated for $nX = 2b, 2c, 3a, 3b$, and $(2a, 2a, 2a, 2a, 2a, 23a)$ -generated;
- $G = Co_1$. Then G is $(nX, nX, 23a)$ -generated for $nX \neq 2a, 2b, 2c, 3a, 3b, 4a$, while it is $(3b, 3b, 26a)$ -generated, $(4a, 4a, 16b)$ -generated, $(nX, nX, nX, 23a)$ -generated for $nX = 2b, 2c$, $(2a, 2a, 2a, 13a)$ -generated, and $(3a, 3a, 3a, 10e)$ -generated.
- $G = J_4$. Then G is $(nX, nX, 43a)$ -generated for $nX \neq 2a, 2b$, and $(nX, nX, nX, 43a)$ -generated for $nX = 2a, 2b$.
- $G = Fi'_{24}$. Then G is $(nX, nX, 29a)$ -generated for $nX \neq 2a, 2b, 3a, 3b$ and $(nX, nX, nX, 29a)$ -generated for $nX = 2a, 2b, 3a, 3b$.
- $G = B$. Then G is $(nX, nX, 47a)$ -generated for $nX \neq 2a, 2b, 2c, 2d$, $(nX, nX, nX, 47a)$ -generated for $nX = 2b, 2c, 2d$, and $(2a, 2a, 2a, 2a, 47a)$ -generated.

3.2. Let us denote by $\alpha_G(nX)$ the minimum number of elements from a given non-trivial class nX required to generate G . Applying Lemma 2.3, we obtain the following estimates:

- $\alpha_G(nX) \geq 3$ for $(G, nX) \in \{(J_2, 3a), (McL, 3a), (Suz, 3a), (Fi_{22}, 3a), (Ly, 3a), (Fi_{23}, 3a), (Fi_{23}, 3b), (Co_1, 3a), (Fi'_{24}, 3a), (Fi'_{24}, 3b)\}$;
- $\alpha_G(2a) \geq 5$ for $G = Fi_{22}, Fi_{23}$.

This means that for these groups and classes the size of a generating set from the class nX given above in 3.1 is the best possible, except possibly for the cases $(Suz, 3a)$ and $(Fi_{22}, 3b)$.

We can also obtain the exact value of $\alpha_G(nX)$ for $(G, nX) \in \{(HS, 4a), (J_2, 2a)\}$. Namely, using MAGMA and the representations available in [2], we get the following:

- for $G = HS$, $\alpha_G(4a) = 3$ (here we have looked at an irreducible representation of G of degree 20 over F_2);
- for $G = J_2$, $\alpha_G(2a) = 4$ (here we have looked at an irreducible representation of G of degree 6 over F_2).

3.3. Finally, let us consider the Monster group M . This group requires a slightly different approach, since not all the information we need is available in GAP (in fact, the maximal subgroups of this group are not yet completely known).

So, let $G = M$. Observe (e.g., see [19]) that there are no maximal subgroups of G containing both elements of order 59 and 71. Computing the structure constants, we obtain the following:

- (1) $\Delta_G(nX, nX, 59a) > 0$ and $\Delta_G(nX, nX, 71a) > 0$ for all the classes $nX \neq 2a, 2b$;
- (2) $\Delta_G(nX, nX, nX, 59a) > 0$ and $\Delta_G(nX, nX, nX, 71a) > 0$ for the classes $nX = 2a, 2b$.

It follows that G can be generated by 3 conjugates from each class $nX \neq 2a, 2b$, and by 5 conjugates from each of the classes $nX = 2a, 2b$. However, in [18], it is shown that 3 suitable conjugates from the class $2b$ can generate G . Furthermore, for the class $2a$, a better bound was obtained by Zisser in [23], namely: $\alpha_M(2a) \leq 4$.

The results obtained above may be summarized in the following:

Theorem 3.1. *Let G be a finite sporadic simple group, and let g be a non-identity element of G . Denote by $\alpha_G(g)$ the minimum number of conjugates of g required to generate G . Then the following holds:*

- (1) *If $G \neq M$ and $g \in G$ is not an involution, then $\alpha_G(g) = 2$ unless:*
 - $(G, g) \in \{(J_2, 3a), (HS, 4a), (McL, 3a), (Ly, 3a), (Co_1, 3a), (Fi_{22}, 3a), (Fi_{23}, 3a), (Fi_{23}, 3b), (Fi'_{24}, 3a), (Fi'_{24}, 3b)\}$. In these cases $\alpha_G(g) = 3$;
 - $(G, g) = (Fi_{22}, 3b)$, in which case $2 \leq \alpha_G(g) \leq 3$;
 - $(G, g) = (Suz, 3a)$, in which case $3 \leq \alpha_G(g) \leq 4$;
- (2) *If $G \neq M$ and $g \in G$ is an involution, then $\alpha_G(g) = 3$ unless:*
 - $(G, g) \in \{(J_2, 2a), (Co_2, 2a), (B, 2a)\}$, in which case $\alpha_G(g) = 4$;
 - $(G, g) \in \{(Fi_{22}, 2a), (Fi_{23}, 2a)\}$. In these cases $5 \leq \alpha_G(g) \leq 6$;
- (3) *If $G = M$ and $g \in G$ is not an involution, then $2 \leq \alpha_G(g) \leq 3$;*
- (4) *If $G = M$ and $g \in G$ is an involution, then $3 \leq \alpha_G(g) \leq 4$.*

4. THE COVERING GROUPS: GENERATION BY CONJUGATES

The covering groups of the simple sporadic groups can be dealt with using the same machinery exploited above. Likewise, the notation (notably for conjugacy classes) is the one fixed in the Introduction, following [2] and [9].

For the reader's sake, the following elementary observation seems to be in order.

Suppose that G is a covering group of the simple group H (that is, G is quasi-simple with $G/Z(G) = H$). Then, if S is any generating set for H , its preimage in G via the natural map is clearly a generating set for G . Obviously, if S consists of elements of the same order, then its preimage in G consists of elements of the same order modulo $Z(G)$.

In view of this, by taking preimages we can transfer the information obtained in the previous section on the generation by conjugates of the sporadic simple groups to their covering groups. The overall results are summarized in the following:

Theorem 4.1. *Let G be a covering group of a finite simple sporadic group and let g be a non-central element of G . Denote by $\alpha_G(g)$ the minimal number of conjugates of g required to generate G , and by \bar{g} the image of g in $\bar{G} = G/Z(G)$. Then the following holds:*

- (1) *If \bar{g} is not an involution, then $\alpha_G(g) = 2$, unless:*

- $(G, g) \in \{(2 \cdot J_2, 3a), (2 \cdot J_2, 6a), (2 \cdot HS, 4b), (2 \cdot HS, 4c), (3 \cdot McL, 3c), (3 \cdot McL, 3d), (3 \cdot McL, 3e), (2 \cdot Fi_{22}, 3a), (2 \cdot Fi_{22}, 6a), (3 \cdot Fi_{22}, 3c), (3 \cdot Fi_{22}, 3d), (3 \cdot Fi_{22}, 3e), (6 \cdot Fi_{22}, 3c), (6 \cdot Fi_{22}, 3d), (6 \cdot Fi_{22}, 3e), (6 \cdot Fi_{22}, 6m), (6 \cdot Fi_{22}, 6n), (6 \cdot Fi_{22}, 6o), (2 \cdot Co_1, 3a), (2 \cdot Co_1, 6a), (3 \cdot Fi'_{24}, 3c), (3 \cdot Fi'_{24}, 3d)\}, (3 \cdot Fi'_{24}, 3e)\}$. In these cases $\alpha_G(g) = 3$;
 - $(G, g) \in \{(2 \cdot Fi_{22}, 3b), (2 \cdot Fi_{22}, 6b), (3 \cdot Fi_{22}, 3f), (6 \cdot Fi_{22}, 3f), (6 \cdot Fi_{22}, 6p)\}$. In these cases $2 \leq \alpha_G(g) \leq 3$;
 - $(G, g) \in \{(2 \cdot Suz, 3a), (2 \cdot Suz, 6a), (3 \cdot Suz, 3c), (3 \cdot Suz, 3d), (3 \cdot Suz, 3e), (6 \cdot Suz, 3c), (6 \cdot Suz, 3d), (6 \cdot Suz, 3e), (6 \cdot Suz, 6g), (6 \cdot Suz, 6h), (6 \cdot Suz, 6i)\}$. In these cases $3 \leq \alpha_G(g) \leq 4$;
- (2) If \bar{g} is an involution, then $\alpha_G(g) = 3$, except for the following cases:
- If $(G, g) \in \{(2 \cdot J_2, 2b), (2 \cdot J_2, 2c), (2 \cdot B, 2b)\}$, then $\alpha_G(g) = 4$;
 - If $(G, g) \in \{(2 \cdot Fi_{22}, 2b), (2 \cdot Fi_{22}, 2c), (3 \cdot Fi_{22}, 2a), (3 \cdot Fi_{22}, 6a), (3 \cdot Fi_{22}, 6b), (6 \cdot Fi_{22}, 2b), (6 \cdot Fi_{22}, 2c), (6 \cdot Fi_{22}, 6c), (6 \cdot Fi_{22}, 6d), (6 \cdot Fi_{22}, 6e), (6 \cdot Fi_{22}, 6f)\}$, then $5 \leq \alpha_G(g) \leq 6$.

5. CYCLIC AND ALMOST CYCLIC ELEMENTS IN THE REPRESENTATIONS OF FINITE SPORADIC GROUPS

In this section we determine the occurrence of cyclic and almost cyclic elements in the representations of the finite sporadic simple groups. For their relevance as well as for technical reasons, we have confined our analysis to the case of elements of prime-power order. The results will be summarized in Theorem 7.1 and Theorem 7.2 (Section 7).

To simplify the notation, if $\Phi : G \rightarrow GL(n, F)$ is a faithful irreducible representation, we will identify G with $\Phi(G)$ (and $\Phi(g)$ with g). Moreover, when we say below that an element of G is almost cyclic, we mean that the element is almost cyclic but not cyclic, and when we say that an element is not almost cyclic, we mean that it is neither cyclic nor almost cyclic. Finally, we denote by d a prime-power integer, and we always assume $d > 2$. Indeed, observe that, in view of Lemma 2.4, no involution of a sporadic simple group can be represented by a cyclic matrix, and hence we will disregard completely generating sets of involutions in our analysis.

In order to apply Lemma 2.1, we fully exploit the results obtained in Section 3 on the generation of G by conjugates. Next, we refer to the paper of Jansen ([13]), giving the minimal degree of the faithful irreducible representations Φ of G , as well as to the work of Hiss and Malle ([12]) on the low-dimensional representations of quasi-simple groups. By Lemma 2.1, we must have $\dim \Phi \leq \alpha_G(g)(|g| - 1)$. If this bound is not met by any Φ , then we are done: $\Phi(g)$ cannot be neither cyclic nor almost cyclic. Otherwise, the list of representations meeting the bound is usually small: if ℓ does not divide the order of g , and the relevant Brauer character tables are known, we get the desired answers using GAP; otherwise, we make use of MAGMA, applying it to the relevant representations as provided by the Atlas on line ([2]).

The results obtained are as follows:

5.1. $G = McL, He, Suz, O'N, J_4, HN, Th, Fi_{22}, Fi_{23}, Fi'_{24}, B$. By Theorem 3.1, it follows from Lemma 2.1 and [13] that for all the listed groups G no element g of prime-power order d can be almost cyclic.

5.2. $G = M_{11}$. In view of [13] and Lemma 2.1, we are left to examine the classes $4a$ and $5a$ only for $\ell = 3$, and the classes $8a, 8b, 11a$ and $11b$ for every ℓ . Since the Brauer character tables are known for any characteristic, using the GAP routines we can answer completely the case when ℓ does not divide d . We obtain that cyclic or almost cyclic elements occur exactly as listed in the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	10	$11a, b$	cyclic
	11	$11a, b$	cyclic
2	10	$11a, b$	cyclic
3	5	$4a$	almost cyclic
	5	$5a; 8a, b; 11a, b$	cyclic
	10	$11a, b$	cyclic
5	10	$11a, b$	cyclic
	11	$11a, b$	cyclic
11	9	$8a, b$	almost cyclic

So, we are left to examine the classes $8a, 8b$ when $\ell = 2$ and the classes $11a, 11b$ when $\ell = 11$. Note that $(8a)^5 \in 8b$ and $(11a)^2 \in 11b$, and therefore for our purposes it is irrelevant whether an element g of order 8 (resp. 11) belongs the class $8a$ or $8b$ (resp. $11a$ or $11b$). Denoting by a and b the 'standard generators' of M_{11} , of order respectively 2 and 4, given in [2], the following holds:

(i) If $\ell = 2$, by [12] the bound given by Lemma 2.1 is only met by a representation Φ of degree 10. Pick $g = bab^2(ab)^3ba$. Then g has order 8, and its invariant factors (computed using MAGMA) are $\{(x-1)^2, (x-1)^8\}$. So g is not almost cyclic.

(ii) If $\ell = 11$, by [12] Φ must have degree 9, 10, 11 or 16. Pick $g = ab$. Then g has order 11. If $\dim \Phi = 9$ or 10 , by Proposition 2.5 g is cyclic. If $\dim \Phi = 11$, g is cyclic by Corollary 2.6. Finally, if $\dim \Phi = 16$, the invariant factors of g are $\{(x-1)^5, (x-1)^{11}\}$. So g is not almost cyclic.

5.3. $G = M_{12}$. In view of [13] and Lemma 2.1, we are left to examine the classes $8a, 8b, 11a$ and $11b$, for every ℓ . Since the Brauer character tables are known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	11	$11a, b$	cyclic
2	10	$11a, b$	cyclic
3	10	$11a, b$	cyclic
5	11	$11a, b$	cyclic

So, we are left to examine the classes $8a, 8b$ when $\ell = 2$ and the classes $11a, 11b$ when $\ell = 11$. Note that $(11a)^2 \in 11b$, and therefore for our purposes it is irrelevant whether

an element g of order 11 belongs the class $11a$ or $11b$. Denoting by a and b the 'standard generators' of M_{12} , of order respectively 2 and 3, given in [2], the following holds:

(i) If $\ell = 2$, by [12] Φ must have degree 10. Constructing this representation using MAGMA, we obtain that, for g in both classes $8a$ and $8b$, the invariant factors are $\{(x-1)^2, (x-1)^8\}$. Thus g is not almost cyclic.

(ii) If $\ell = 11$, by [12] Φ must have either degree 11 (there are two such representations) or degree 16. Pick $g = ab$. Then g has order 11. If $\dim \Phi = 11$, then g is cyclic by Corollary 2.6. If $\dim \Phi = 16$, the invariant factors of g are $\{(x-1)^5, (x-1)^{11}\}$. So g is not almost cyclic.

5.4. $G = J_1$. In view of [13] and Lemma 2.1, we only need to examine the classes $11a$, $19a$, $19b$ and $19c$ when $\ell = 2$, the classes $19a$, $19b$ and $19c$ when $\ell = 7, 19$, and the classes $5a$, $5b$, $7a$, $11a$, $19a$, $19b$ and $19c$ when $\ell = 11$. The Brauer character tables being known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
2	20	$19a, b, c$	almost cyclic
11	7	$7a; 19a, b, c$	cyclic
	14	$19a, b, c$	almost cyclic

So, we are left to examine the class $11a$ for $\ell = 11$ and the class $19a$ for $\ell = 19$ (note that $(19a)^2 \in 19b$ and $(19b)^2 \in 19c$). Denoting by a and b the 'standard generators' of M_{12} , of order respectively 2 and 3, given in [2], the following holds:

(i) If $\ell = 11$, by [12] Φ must have either degree 7 or degree 14. Let us pick $g = babab^2abab^2ab^2(ab)^3ab^2ab$. Then g has order 11. If $\dim \Phi = 7$, then g is cyclic by Proposition 2.5. If $\dim \Phi = 14$, the invariant factors of g are $\{(x-1)^3, (x-1)^{11}\}$, and hence g is not almost cyclic.

(ii) If $\ell = 19$, by [12] Φ must have either degree 22 or degree 34. Pick $g = abab^2$. Then g has order 19. If $\dim \Phi = 22$, the invariant factors of g are $\{(x-1)^3, (x-1)^{19}\}$. If $\dim \Phi = 34$, the invariant factors of g are $\{(x-1)^{15}, (x-1)^{19}\}$. Thus, in both cases, g is not almost cyclic.

5.5. $G = M_{22}$. In view of [13] and Lemma 2.1, we only need to examine the classes $7a$, $7b$, $8a$, $11a$ and $11b$ when $\ell = 2$, and the classes $11a$ and $11b$ when $\ell = 11$. Also note that $(11a)^2 \in 11b$.

Denoting by a and b the 'standard generators' of M_{12} , of order respectively 2 and 4, given in [2], the following holds:

(i) If $\ell = 2$, by [12] Φ must have degree 10 (there are two such representations). By inspection of the Brauer character tables, we see that if g has order 11, then g is cyclic, whereas if g has order 7, then g is not almost cyclic. Next, pick $g = bab^2abab^2ab^2ab^2aba$. Then g has order 8 and, using MAGMA, we get that the invariant factors of g are $\{(x-1)^2, (x-1)^8\}$. Thus g is not almost cyclic.

(ii) If $\ell = 11$, by [12] Φ must have degree 20. Let $g = ab$. Then g has order 11. In this case, the invariant factors of g are $\{(x-1)^9, (x-1)^{11}\}$. Thus g is not almost cyclic.

5.6. $G = J_2$. In view of [13] and Lemma 2.1, we only need to examine the class $8a$ when $\ell \neq 2$ and the classes $3a, 4a, 5a, 5b, 5c, 5d, 7a$ and $8a$ when $\ell = 2$.

Denoting by a and b the 'standard generators' of M_{12} , of order respectively 2 and 3, given in [2], the following holds:

(i) By inspection of the Brauer character tables, whenever ℓ does not divide d , there is only one instance in which a cyclic or almost cyclic element can occur, namely the following:

ℓ	$\dim \Phi$	dX	type
2	6	$7a$	cyclic

(ii) If $\ell = 2$, by [12] Φ must have either degree 6 or degree 14. Moreover, $g_4 = (ab^2ab)^3$ has order 4 and $g_8 = (abab^2)^2(ab)^3bab^2ab$ has order 8.

Let $\dim \Phi = 6$ (there are two such representations). Then the invariant factors of g_4 are $\{(x-1)^3, (x-1)^3\}$, and hence g_4 is not almost cyclic. On the other hand, the minimal and characteristic polynomial of g_8 coincide. So g_8 is cyclic.

Let $\dim \Phi = 14$ (there are two such representations). In view of Lemma 2.1, we only need to deal with g_8 . As the invariant factors of g_8 are $\{(x-1)^6, (x-1)^8\}$, g_8 is not almost cyclic.

5.7. $G = M_{23}$. In view of [13] and Lemma 2.1, we only need to examine the classes $23a$ and $23b$ when $\ell \neq 2$, and the classes $7a, 7b, 8a, 11a, 11b, 23a$ and $23b$, when $\ell = 2$. Since the Brauer character tables are known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	22	$23a, b$	cyclic
2	11	$11a, b; 23a, b$	cyclic
3	22	$23a, b$	cyclic
5	22	$23a, b$	cyclic
7	22	$23a, b$	cyclic
11	22	$23a, b$	cyclic

Let us denote by a and b the 'standard generators' of M_{23} , of order respectively 2 and 4, given in [2].

Let $\ell = 2$. By [12] Φ must have degree 11. Pick $g = (ab)^4b(ab)^2bab^2$. Then g has order 8, and the invariant factors of g are $\{(x-1)^3, (x-1)^8\}$. So g is not almost cyclic.

Let $\ell = 23$. By [12] Φ must have degree 21. In this case, an element of order 23 is cyclic by Proposition 2.5.

5.8. $G = HS$. Arguing as above, we have only to examine the classes $11a$ and $11b$ when $\ell = 2$. These elements turn out not to be almost cyclic.

5.9. $G = J_3$. The only classes to be examined are $17a, 17b, 19a$ and $19b$ when $\ell = 3$. By [12] Φ must have degree 18 (there are two such representations). Using as above the GAP routines, it turns out that the elements of order 17 are almost cyclic, while the elements of order 19 are cyclic.

5.10. $G = M_{24}$. The only classes to be examined are the classes $23a$ and $23b$ when $\ell \neq 2$, and the classes $7a$, $7b$, $8a$, $11a$, $23a$ and $23b$ when $\ell = 2$. Since the Brauer character tables are known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	23	$23a, b$	cyclic
2	11	$11a; 23a, b$	cyclic
3	22	$23a, b$	cyclic
5	23	$23a, b$	cyclic
7	23	$23a, b$	cyclic
11	23	$23a, b$	cyclic

Let us denote by a and b the 'standard generators' of M_{24} , of order respectively 2 and 4, given in [2].

If $\ell = 2$, by [12] Φ must have degree 11 (there are two such representations). Pick $g = (ba)^2(b^2a)^2b$. Then g has order 8, and the invariant factors of g are $\{(x-1)^3, (x-1)^8\}$. So g is not almost cyclic.

If $\ell = 23$, by [12] Φ must have degree 23. In this case, the elements of order 23 are cyclic by Corollary 2.6.

5.11. $G = Ru$. The only classes to be examined are $16a$, $16b$, $29a$ and $29b$, when $\ell = 2$. Note that $(16a)^3 \in (16b)$. By [12], Φ must be of degree 28. A GAP computation shows that the elements of order 29 are cyclic. Now, denote by a and b the 'standard generators' of Ru , of order respectively 2 and 4, given in [2]. Pick $g = (ba)^2b^2ab^2(ab)^3bab(ab^2)^5(ab)^2b(ab)^2$. Then g has order 16, and it is not almost cyclic, since its invariant factors are $\{(x-1)^{12}, (x-1)^{16}\}$.

5.12. $G = Co_3$. The only classes to be examined are the classes $23a$ and $23b$, for every ℓ . Since the Brauer character tables are known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	23	$23a, b$	cyclic
2	22	$23a, b$	cyclic
3	22	$23a, b$	cyclic
5	23	$23a, b$	cyclic
7	23	$23a, b$	cyclic
11	23	$23a, b$	cyclic

If $\ell = 23$, by [12] Φ must have degree 23. Hence the elements of order 23 are cyclic by Corollary 2.6.

5.13. $G = Co_2$. The only classes to be examined are the classes $16a$, $16b$, $23a$ and $23b$, for every ℓ . Since the Brauer character tables are known for any characteristic, using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	23	$23a, b$	cyclic
2	22	$23a, b$	cyclic
3	23	$23a, b$	cyclic
5	23	$23a, b$	cyclic
7	23	$23a, b$	cyclic
11	23	$23a, b$	cyclic

If $\ell = 2$, we need to examine the classes $16a$ and $16b$. By [12], Φ must have degree 22. Using MAGMA, we can check that, for both classes, the invariant factors are $\{(x-1)^8, (x-1)^{14}\}$. So, these elements are not almost cyclic.

If $\ell = 23$, by [12] Φ must have degree 23, and the elements of order 23 are cyclic by Corollary 2.6.

5.14. $G = Ly$. G is $(dX, dX, 67a)$ -generated for all the classes $dX \neq 3a$, while it is $(3a, 3a, 67a)$ -generated (three being the minimal size of a generating set from the class $3a$). The only classes to be examined are the classes $67a$, $67b$ and $67c$, when $\ell = 5$. Observe that $(67a)^2 \in 67b$ and $(67a)^7 \in 67c$, and therefore for our purposes it is irrelevant whether an element g of order 67 belongs to one or another class. Denote by a and b the 'standard generators' of Ly , of order respectively 2 and 5, given in [2]. According to [12], Φ must have degree 111, and moreover is unique (unpublished work of Lux and Ryba). Let us pick $g = (ab)^3b$. Then g has order 67, but it is not almost-cyclic, since its minimum polynomial is $m_g(x) = x^{67} - 1$, whereas its characteristic polynomial is $p_g(x) = (x^{67} - 1)(x^{22} + x^{20} - x^{18} + 2x^{17} - x^{16} - x^{15} + 2x^{14} + x^{12} + x^{10} + 2x^8 - x^7 - x^6 + 2x^5 - x^4 + x^2 + 1)(x^{22} - x^{21} + 3x^{20} + 2x^{19} - x^{18} + 2x^{15} + 2x^{14} - x^{12} + 3x^{11} - x^{10} + 2x^8 + 2x^7 - x^4 + 2x^3 + 3x^2 - x + 1)$.

5.15. $G = Co_1$. The only classes to be examined are the classes $13a$, $16a$, $16b$, $23a$ and $23b$, when $\ell = 2$. Also, note that $(23a)^5 \in 23b$. Let us denote by a and b the 'standard generators' of Co_1 , of order respectively 2 and 3, given in [2]. According to [12], Φ must have degree 24. Moreover, such a Φ is unique (since a proof of this fact is not available in the literature, we have checked it independently. See Appendix). Pick $g = (ba)^2b^2(ab)^2(ba)^4b(ba)^3$. Then g has order 23 and it is almost cyclic, since it has minimum polynomial $m_g(x) = x^{23} - 1$ and characteristic polynomial $p_g(x) = (x-1)(x^{23} - 1)$. Next, let $g = (b(abab^2)^2ab^2ab(ab^2)^5abab^2(ab)^2)^2$. Then g has order 13, its minimal polynomial is $m_g(x) = \frac{x^{13}-1}{x-1}$ and its characteristic polynomial is $p_g(x) = (m_g(x))^2$. So g is not almost cyclic.

Finally, the elements of order 16 cannot be almost cyclic. Indeed, assume that $g \in G$ of order 16 is such that $\Phi(g)$ is almost cyclic. Observe that both classes $16a$ and $16b$ have non-trivial intersection with a maximal subgroup H of type Co_2 . Since the minimal degree of an irreducible representation of Co_2 is 22, $\Phi|_H = 2\Psi_1 + \Psi_{22}$, where Ψ_i are irreducible representations of H of degree i . This means that, considering g as an element of Co_2 , $\Psi_{22}(g)$ should be almost cyclic. But we have already proved that this cannot happen.

5.16. $G = M$. By Theorem 3.1, G can be generated by at most 3 conjugates from each class dX . Since $\dim \Phi \geq 196882$ by [13]), by Lemma 2.1 no element g of prime-power order d can be almost cyclic.

6. CYCLIC AND ALMOST CYCLIC ELEMENTS IN THE REPRESENTATIONS OF THE COVERING GROUPS

The covering groups of the simple sporadic groups can be dealt with using the same machinery exploited above. As in the previous section, for technical reasons, we confine our analysis to the case of elements of prime-power order (modulo the centre) which can be represented by cyclic or almost cyclic matrices in faithful irreducible representations. The notation (notably for conjugacy classes) is the one fixed in the Introduction, following [2] and [9].

For the reader's sake, the following elementary observations seem to be in order:

1) An element x of G has prime-power order modulo $Z(G)$ if and only if x is the product of a central element by an element, say x_1 , of G of prime-power order. Obviously, for any F -representation Φ of G , $\Phi(x)$ is cyclic (almost cyclic) if and only if $\Phi(x_1)$ is cyclic (almost cyclic).

2) Let Φ be an F -representation of G and let $\tilde{\Phi}$ be the associated projective representation of its simple central quotient H (defined by $\tilde{\Phi}(Z(G)x) = \Phi(x)$ for $x \in G$). Suppose that x has order two modulo the centre, and $\Phi(x)$ is cyclic (almost cyclic). Then $\Phi(x)$, and so also $\tilde{\Phi}(Z(G)x)$, is a pseudoreflection. But this contradicts Lemma 2.4.

We will fully exploit the results on generation by conjugates obtained in Sections 3 and 4. Furthermore, in view of 1) and 2), we will only have to deal with the conjugacy classes of the covering group G which consist of elements of prime-power order d whose images in H have order greater than two. Therefore, from now on, the notation dX only refers to such classes.

We obtain the following results:

6.1. $G = 6 \cdot M_{22}, 12 \cdot M_{22}, 2 \cdot HS, 3 \cdot McL, 2 \cdot Fi_{22}, 6 \cdot Fi_{22}, 3 \cdot Fi'_{24}, 2 \cdot B$. By Theorem 4.1, it follows from Lemma 2.1 and [13] that no $g \in G$ belonging to any of the classes dX can be almost cyclic.

6.2. $G = 2 \cdot M_{12}$. G can be generated by two conjugates from any of the classes dX .

In view of [13] and Lemma 2.1, by inspecting the Brauer character tables and using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	10	$11a, b$	cyclic
	12	$11a, b$	almost cyclic
3	6	$5a$	almost cyclic
	6	$8a, b, c, d; 11a, b$	cyclic
	10	$11a, b$	cyclic
5	10	$11a, b$	cyclic
	12	$11a, b$	almost cyclic

Again by [13] and Lemma 2.1, we may rule out the class $3a$ for $\ell = 3$ and the class $5a$ for $\ell = 5$. Thus, we are only left to examine the classes $11a$ and $11b$ for $\ell = 11$.

Now, if $\ell = 11$, by [12] Φ must have either degree 10 (there are two such representations) or degree 12. Let g be an element of order 11. If $\dim \Phi = 10$, by Proposition 2.5 $\Phi(g)$ is cyclic. If $\dim \Phi = 12$, then $\Phi(g)$ is almost cyclic by Corollary 2.6.

6.3. $G = 2 \cdot M_{22}$. G can be generated by two conjugates from any of the classes dX .

In view of [13] and Lemma 2.1, by inspecting the Brauer character tables and using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	10	$11a, b$	cyclic
3	10	$11a, b$	cyclic
5	10	$11a, b$	cyclic
7	10	$11a, b$	cyclic

Thus, we are left to examine the classes $7a$ and $7b$ for $\ell = 7$ and the classes $11a$ and $11b$ for $\ell = 11$.

Let us denote by a and b the 'standard generators' of $2 \cdot M_{22}$, of order respectively 2 and 4, given in [2].

Let $\ell = 7$. By [12] Φ must have degree 10. Also, note that $(7a)^3 \in 7b$. Pick $g = abab^3ab^2abab^3ab^2a$. Then g has order 7, and its invariant factors are $\{(x-1)^3, (x-1)^7\}$. So g is not almost cyclic.

Let $\ell = 11$. By [12] Φ must have degree 10 (there are two such representations). If $g \in G$ has order 11, then g is cyclic by Proposition 2.5.

6.4. $G = 3 \cdot M_{22}$. G can be generated by two conjugates from any of the classes dX .

In view of [13] and Lemma 2.1, by inspecting the Brauer character tables and using the GAP routines we obtain that, whenever ℓ does not divide d , cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
2	6	$5a$	almost cyclic
	6	$7a, b; 11a, b$	cyclic

Again by [13] and Lemma 2.1, we are left to examine only the classes $4a$, $4b$ and $8a$ for $\ell = 2$. Let us denote by a and b the 'standard generators' of $3 \cdot M_{22}$, of order respectively 2 and 4, given in [2].

If $\ell = 2$, by [12] Φ must have degree 6 (there are two such representations). Using MAGMA, we see that the invariant factors of the elements of order 4 are either $\{(x-1)^3, (x-1)^3\}$ or $\{(x-1)^2, (x-1)^4\}$. So these elements are not almost cyclic. On the other hand, for an element of order 8 the minimum polynomial and the characteristic polynomial coincide. So, the element is cyclic.

6.5. $G = 4 \cdot M_{22}$. G can be generated by two conjugates from any of the classes dX . By Lemma 2.1 and [13], we only have to examine the classes $11a$ and $11b$ for $\ell = 7$. Inspection of the Brauer character tables shows that these elements are not almost cyclic.

6.6. $G = 2J_2$. G can be generated by two conjugates from any of the classes $dX \neq 3a$; while it can be generated by three conjugates from the class $dX = 3a$.

Taking into account [13] and Lemma 2.1, we obtain the following:

(i) Whenever ℓ does not divide d , the Brauer character tables, via the GAP routines, show that cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	6	$7a; 8a, b$	cyclic
3	6	$7a; 8a, b$	cyclic
5	6	$7a; 8a, b$	cyclic
7	6	$8a, b$	cyclic

(ii) We are left to examine the class $3a$ for $\ell = 3$, the classes $5a$ and $5b$ for $\ell = 5$ and the class $7a$ for $\ell = 7$.

If $\ell = 3$, by [12] Φ must have degree 6 (there are two such representations). However, using MAGMA, we see that the elements of order 3 have as invariant factors either $\{(x-1)^2, (x-1)^2\}$, $\{(x-1)^2, (x-1)^3\}$ or $\{(x-1)^3, (x-1)^3\}$.

If $\ell = 5$, by [12] Φ must have degree 6. Using MAGMA, we see that the elements of order 5 have as invariant factors either $\{(x-1)^3, (x-1)^3\}$ or $\{(x-1)^2, (x-1)^4\}$.

If $\ell = 7$, by [12] Φ must have degree 6 (there are two such representations). By Proposition 2.5, the elements of order 7 are cyclic.

6.7. $G = 2Suz$. G can be generated by two conjugates from any of the classes $dX \neq 3a$, and by four conjugates from the class $3a$.

In view of [13] and Lemma 2.1, we need to examine only the case $\ell = 3$.

If 3 is coprime to d , inspection of the Brauer character table produces the following single occurrence:

ℓ	$\dim \Phi$	dX	type
3	12	$13a, b$	cyclic

Since, by [12], Φ must have degree 12, we are left to examine only the classes $9a$ and $9b$ for $\ell = 3$. Also, note that $(9a)^2 \in 9b$. Let us denote by a and b the 'standard generators' of $2Suz$, of order respectively 4 and 3, given in [2]. Pick $g = (ab)^3 ab^2 ab(ab^2)^5 ab$. Then g has order 9, and its invariant factors are $\{(x-1)^4, (x-1)^8\}$. So g is not almost cyclic.

6.8. $G = 3Suz$. G can be generated by two conjugates from any of the classes $dX \neq 3c, 3d, 3e$, and by four conjugates from any of the classes $dX = 3c, 3d, 3e$. In view of [13] and Lemma 2.1, we need to examine only the case $\ell = 2$.

If 2 is coprime to d , inspection of the Brauer character table produces the following occurrences:

ℓ	$\dim \Phi$	dX	type
2	12	$11a$	almost cyclic
	12	$13a, b$	cyclic

Since, by [12], Φ must have degree 12 (there are two such representations), we are left to examine only the classes $8a, 8b$ and $8c$. Using MAGMA, we see that the invariant factors of an element g of order 8 are either $\{(x-1)^6, (x-1)^6\}$, or $\{(x-1)^5, (x-1)^7\}$, or $\{(x-1)^4, (x-1)^8\}$. Hence g is not almost cyclic.

6.9. $G = 6 \cdot \text{Suz}$. G can be generated by two conjugates from any of the classes $dX \neq 3c, 3d, 3e$, and by four conjugates from any of the classes $dX = 3c, 3d, 3e$.

Taking into account [13] and Lemma 2.1, we obtain the following:

(i) Whenever ℓ does not divide d , the Brauer character tables, via the GAP routines, show that cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	12	11a	almost cyclic
	12	13a, b	cyclic
5	12	11a	almost cyclic
	12	13a, b	cyclic
7	12	11a	almost cyclic
	12	13a, b	cyclic
11	12	13a, b	cyclic
13	12	11a	almost cyclic

(ii) We are left to examine only the class $7a$ for $\ell = 7$, the class $11a$ for $\ell = 11$ and the classes $13a$ and $13b$ for $\ell = 13$.

Let us denote by a and b the 'standard generators' of $6 \cdot \text{Suz}$, of order respectively 4 and 3, given in [2].

If $\ell = 7$, by [12] Φ must have degree 12 (two representations). Pick $g = ((ab)^3 a^2 bab)^6$. Then g has order 7 and its invariant factors are $\{(x-1)^6, (x-1)^6\}$. So g is not almost cyclic.

If $\ell = 11$, by [12] Φ must have degree 12 (two representations). Thus, an element g of order 11 is almost cyclic, by Corollary 2.6.

If $\ell = 13$, by [12] Φ must have degree 12 (two representations). Thus, an element g of order 13 is cyclic, by Proposition 2.5.

6.10. $G = 3 \cdot J_3$. G can be generated by two conjugates from any of the classes dX .

By [12], either $\dim \Phi = 9$, or $\dim \Phi = 18$, or $\dim \Phi \geq 126$. In the latter case, in view of Lemma 2.1 we get a contradiction (for all the classes dX). So, we may assume that either $\dim \Phi = 9$ or $\dim \Phi = 18$. Taking into account [13] and Lemma 2.1, we obtain the following:

(i) Whenever ℓ does not divide d , the Brauer character tables, via the GAP routines, show that cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	18	$17a, b$	almost cyclic
	18	$19a, b$	cyclic
2	9	$9a, b, c; 17a, b; 19a, b$	cyclic
	18	$17a, b$	almost cyclic
	18	$19a, b$	cyclic
5	18	$17a, b$	almost cyclic
	18	$19a, b$	cyclic
17	18	$19a, b$	cyclic
19	18	$17a, b$	almost cyclic

(ii) We are left to examine only the elements of order 8 for $\ell = 2$, of order 5 for $\ell = 5$, of order 17 for $\ell = 17$ and of order 19 for $\ell = 19$.

Let us denote by a and b the 'standard generators' of $3 \cdot J_3$, of order respectively 2 and 3, given in [2]. Then:

(1) If $\ell = 2$, let $g = ((ab^2)^4 ab)^3$. Then g has order 8. By Lemma 2.1, $\dim \Phi = 9$ (there are two such representations). The invariant factors of g are $\{(x-1), (x-1)^8\}$, so g_8 is almost cyclic.

(2) If $\ell = 5$, the elements of order 5 are not almost cyclic by Lemma 2.1.

(3) If $\ell = 17$, by [13] Φ must have degree 18 (there are four of these Φ 's), and if g is an element of order 17, then g is almost cyclic by Corollary 2.6.

(4) If $\ell = 19$, by [13] Φ must have degree 18 (there are four of these Φ 's), and the elements of order 19 are cyclic by Corollary 2.6.

6.11. $G = 2 \cdot Ru$. G can be generated by two conjugates from any of the classes dX .

By [12], either $\dim \Phi = 28$ or $\dim \Phi > 250$. In the latter case, by Lemma 2.1 no $g \in G$ belonging to any of the classes dX can be almost cyclic. So, we may assume that $\dim \Phi = 28$ (there are two such representations), and we are only left to examine the elements of order 29. A computation using GAP if $\ell \neq 29$, and Corollary 2.6 if $\ell = 29$, show that these elements are cyclic.

6.12. $G = 3 \cdot O'N$. G can be generated by two conjugates from any of the classes dX .

If $\ell \neq 7$, by [12] $\dim \Phi \geq 153$, and by Lemma 2.1 no $g \in G$ belonging to any of the classes dX can be almost cyclic. Again by [12], if $\ell = 7$ then either $\dim \Phi = 45$ (there are two representations of this degree), or $\dim \Phi > 250$. As above, the latter case is ruled out by Lemma 2.1. In the former case, $g \in G$ cannot be almost cyclic unless it belongs to one of the classes $31a$ and $31b$. However, for all these elements, since ℓ does not divide their order, we can inspect the Brauer character tables. They show that none of them can be almost cyclic.

6.13. $G = 3 \cdot Fi_{22}$. G can be generated by two conjugates from any of the classes $dX \neq 3c, 3d, 3e, 3f$, and can be generated by three conjugates from any of the classes $dX = 3c, 3d, 3e, 3f$.

Now, by [12], if $\ell \neq 2$ $\dim \Phi \geq 351$. Thus, this case is ruled out by Lemma 2.1. If $\ell = 2$, then either $\dim \Phi = 27$ or $\dim \Phi > 250$. Again, the latter case is ruled out by Lemma 2.1.

In the former case, $g \in G$ cannot be almost cyclic unless it belongs to one of the classes $16a$ and $16b$.

So, assume that $\ell = 2$ and $\dim \Phi = 27$ (there are exactly two such Φ 's (see [14])). Let us denote by a and b the 'standard generators' of $3 \cdot Fi_{22}$, of order respectively 2 and 13, given in [2]. Note that $(16a)^5 \in 16b$, and pick $g = ((ba)^2 b^2 a)^3$. Then g has order 16, and its invariant factors are $\{(x-1), (x-1)^{10}, (x^{16}-1)\}$. So g is not almost cyclic.

6.14. $G = 2 \cdot Co_1$. Here G is generated by two conjugates from any of the classes dX , except for the class $3a$, in which case three conjugate generators suffice.

By [12], either $\dim \Phi = 24$ or $\dim \Phi > 250$. The latter case is ruled out by Lemma 2.1. So, assume that $\dim \Phi = 24$ (such a representation only occurs if $\ell \neq 2$).

Taking into account [13] and Lemma 2.1, we obtain the following:

(i) Whenever ℓ does not divide d and $\ell \neq 3, 5$, the Brauer character tables, via the GAP routines, show that cyclic or almost cyclic elements occur exactly according to the following table:

ℓ	$\dim \Phi$	dX	type
$\ell \nmid G $	24	$23a, b$	almost cyclic
7	24	$23a, b$	almost cyclic
11	24	$23a, b$	almost cyclic
13	24	$23a, b$	almost cyclic

(ii) If $\ell = 3$ or 5 , GAP does not give information on the relevant Brauer characters. However, we may still obtain the desired answers.

First of all observe that, in view of Lemma 2.1, we only need to examine elements of G of order 13, 16 and 23. Furthermore, G has three classes of elements of order 16: a class $16a$ whose elements have centralizer of order 2^6 and two classes $16b$ and $16c$ whose elements have centralizers of order 2^7 . Also, note that $(23a)^5 \in 23b$.

Now, let us denote by a and b the 'standard generators' of $2 \cdot Co_1$, of order respectively 4 and 3, given in [2]. The following holds:

(a) Set $g_{13} = (b^2 ab^2 a^3 bab^2 ab^2 abab^2 aba^3 ba)^3$ and $g_{23} = (ba)^2 b^2 a^3 b^2 ab^2 a^2 (ab^2)^3 abab^2 aba^3 ba$. Then each g_i has exactly order i , and we see that the invariant factors of g_{13} are $\{(x-1)^{12}, (x-1)^{12}\}$ and those of g_{23} are $\{(x-1), (x-1)^{23}\}$. So g_{23} is almost cyclic, whereas g_{13} is not.

(b) The element $g = (ab)^5 (ab^2)(ab)$ has order 16 and its centralizer has order 2^6 , so it belongs to the class $16a$. This element is not almost cyclic, since it has minimum polynomial $m_g(x) = \frac{x^{16}-1}{x^2+1}$ and characteristic polynomial $p_g(x) = \frac{(m_g(x))^2}{x^4+1}$. Next, let $g = (bab^2 a)^3 (ba)^4 ba^2 (ba)^2 b^2 ab (ab^2)^3 aba (ab)^5 ab^2 ab$. It can be checked that both g and $a^2 g$ have order 16 and centralizer of order 2^7 . Moreover, they are not conjugate to each other; hence, they are representatives of the classes $16b$ and $16c$. Neither of them is almost cyclic, since for both of them the minimum polynomial is $m(x) = \frac{x^{16}-1}{x+1}$, whereas the characteristic polynomial is $m(x)(x-1)(x^8+1)$.

(iii) We are now left to examine the cases $\ell = d$ for $\ell = 13$ and 23 . We get the following:

If $\ell = 13$, the invariant factors of g_{13} are $\{(x-1)^{12}, (x-1)^{12}\}$. So this element is not almost cyclic.

If $\ell = 23$: the elements of order 23 are almost cyclic by Corollary 2.6.

7. CYCLIC AND ALMOST CYCLIC ELEMENTS IN THE REPRESENTATIONS OF SPORADIC GROUPS: THE RESULTS

The results that we have obtained are assembled in the following Theorems, whose proof is embodied in the analysis carried out in Sections 5 and 6:

Theorem 7.1. *Let G be a quasi-simple finite sporadic group, F be an algebraically closed field of characteristic ℓ and Φ be an irreducible faithful representation of G over F . Let g be an element of G and suppose that $g = g_1 z$, where $g_1 \in G$ has prime-power order $d > 1$ and $z \in Z(G)$. Denote by dX the conjugacy class of g_1 . Then $\Phi(g)$ is cyclic if and only if one of the cases listed in the following table occurs:*

G	$\dim \Phi$	dX	ℓ
M_{11}	5	$5a; 8a, b; 11a, b$	3
	9	$11a, b$	11
	10	$11a, b$	<i>any</i>
	11	$11a, b$	$\neq 2, 3$
M_{12}	10	$11a, b$	2, 3
	11	$11a, b$	$\neq 2, 3$
$2 \cdot M_{12}$	6	$8a, b, c, d; 11a, b$	3
	10	$11a, b$	$\neq 2$
M_{22}	10	$11a, b$	2
$2 \cdot M_{22}$	10	$11a, b$	$\neq 2$
$3 \cdot M_{22}$	6	$7a, b; 8a; 11a, b$	2
M_{23}	11	$11a, b; 23a, b$	2
	21	$23a, b$	23
	22	$23a, b$	$\neq 2, 23$
M_{24}	11	$11a; 23a, b$	2
	22	$23a, b$	3
	23	$23a, b$	$\neq 2, 3$
J_1	7	$7a; 11a; 19a, b, c$	11
J_2	6	$7a; 8a$	2
$2 \cdot J_2$	6	$7a; 8a, b$	$\neq 2$
J_3	18	$19a, b$	3
$3 \cdot J_3$	9	$9a, b, c; 17a, b; 19a, b$	2
	18	$19a, b$	$\neq 3$
Ru	28	$29a, b$	2
$2 \cdot Ru$	28	$29a, b$	$\neq 2$
$2 \cdot Suz$	12	$13a, b$	3
$3 \cdot Suz$	12	$13a, b$	2
$6 \cdot Suz$	12	$13a, b$	$\neq 2, 3$
Co_3	22	$23a, b$	2, 3
	23	$23a, b$	$\neq 2, 3$
Co_2	22	$23a, b$	2
	23	$23a, b$	$\neq 2$

Theorem 7.2. *Let G be a quasi-simple finite sporadic group, F be an algebraically closed field of characteristic ℓ and Φ be an irreducible faithful representation of G over F . Let g be an element of G and suppose that $g = g_1 z$, where $g_1 \in G$ has prime-power order $d > 1$ and $z \in Z(G)$. Denote by dX the conjugacy class of g_1 . Then $\Phi(g)$ is almost cyclic, but not cyclic, if and only if one of the cases listed in the following table occurs:*

G	$\dim \Phi$	dX	ℓ
M_{11}	5	$4a$	3
	9	$8a, b$	11
$2 \cdot M_{12}$	6	$5a$	3
	12	$11a, b$	$\neq 2, 3$
$3 \cdot M_{22}$	6	$5a$	2
J_1	14	$19a, b, c$	11
	20	$19a, b, c$	2
J_3	18	$17a, b$	3
$3 \cdot J_3$	9	$8a$	2
	18	$17a, b$	$\neq 3$
$3 \cdot Suz$	12	$11a$	2
$6 \cdot Suz$	12	$11a$	$\neq 2, 3$
Co_1	24	$23a, b$	2
$2 \cdot Co_1$	24	$23a, b$	$\neq 2$

APPENDIX

It is well known that the group $G = Co_1$ has an irreducible representation Φ of degree 24 in characteristic 2. However, it seems that the uniqueness of such a representation has not yet been settled in the existing literature. Here we prove the uniqueness of such a Φ working out its restrictions to certain maximal subgroups H_i 's of G , whose Brauer character table is known for $\ell = 2$. In this way, we show that G has a unique irreducible Brauer character ϕ of degree 24 for $\ell = 2$. We keep the notation of GAP for subgroups, irreducible Brauer characters and classes.

Let us first consider the maximal subgroup $H_1 = Co_2$ and its irreducible Brauer characters for $\ell = 2$. This group has (obviously) a unique character χ_1 of degree 1 and a unique character χ_2 of degree 22, while the other characters have degree greater than 24. So, necessarily, $\phi|_{H_1} = 2 \cdot \chi_1 + \chi_2$.

Next, consider $H_2 = 3 \cdot Suz.2$. This group has a unique character χ_1 of degree 1 and a unique character χ_{14} of degree 24, while the other characters have degree greater than 24. So, necessarily, $\phi|_{H_2} = \chi_{14}$.

Using the previous two subgroups we can determine the value of ϕ on the classes $7a$ and $7b$ of G (since H_1 contains elements of the class $7b$ and H_2 contains elements of the class $7a$). We get that $\phi(7a) = -4$ and $\phi(7b) = 3$. These values will be used in the following.

Now, consider the subgroup $H_6 = U_6(2).3.2$. This group has a unique character χ_1 of degree 1, a unique character χ_2 of degree 2 and a unique character χ_3 of degree 20, while the other characters have degree greater than 24. Furthermore, it contains elements of the class $7b$ of G (labelled $7a$ in H_6) for which $\chi_1(7a) = 1$, $\chi_2(7a) = 2$ and $\chi_3(7a) = -1$. So, χ_3 must be a component of $\phi|_{H_6}$. We have three possible decompositions for $\phi|_{H_6}$: $4 \cdot \chi_1 + \chi_3$, $2 \cdot \chi_1 + \chi_2 + \chi_3$ or $2 \cdot \chi_2 + \chi_3$. Looking at the classes $3a$, $3c$ and $3e$ of H_6 , we have a priori the following possibilities:

$\phi _{H_6}$	$3a$	$3c$	$3e$
$4 \cdot \chi_1 + \chi_3$	6	6	12
$2 \cdot \chi_1 + \chi_2 + \chi_3$	6	6	9
$2 \cdot \chi_2 + \chi_3$	6	6	6

However, the classes $3a$, $3c$ and $3e$ of H_6 are fused into the class $3b$ of G . This forces $\phi|_{H_6} = 2 \cdot \chi_2 + \chi_3$.

Next, consider the subgroup $H_7 = (A_4 \times G_2(4)) : 2$. This group has a unique character χ_1 of degree 1, a unique character χ_{11} of degree 2, and three characters χ_2 , χ_{12} , χ_{13} of degree 12, while all the other characters have degree greater than 24. Moreover, this subgroup contains elements belonging to the class $7a$ of G (also labelled $7a$ in H_7) for which $\chi_1(7a) = 1$, $\chi_{11}(7a) = 2$ and $\chi_2(7a) = \chi_{12}(7a) = \chi_{13}(7a) = -2$. So, necessarily, $\phi|_{H_7}$ can only have components of degree 12, and hence 6 possible distinct decompositions.

Looking at the classes $3a$, $3c$, $15e$ and $15f$ of H_7 , we have a priori the following possibilities:

$\phi _{H_7}$	$3a$	$3c$	$15e$	$15f$
$\chi_2 + \chi_{12}$	-12	6	$(-3 - i\sqrt{15})/2$	$(-3 + i\sqrt{15})/2$
$\chi_2 + \chi_{13}$	-12	6	$(-3 + i\sqrt{15})/2$	$(-3 - i\sqrt{15})/2$
$\chi_{12} + \chi_{13}$	-12	-12	3	3
$2 \cdot \chi_2$	-12	24	-6	-6
$2 \cdot \chi_{12}$	-12	-12	$3 - i\sqrt{15}$	$3 + i\sqrt{15}$
$2 \cdot \chi_{13}$	-12	-12	$3 + i\sqrt{15}$	$3 - i\sqrt{15}$

However, the classes $3a$ and $3c$ of H_7 are fused into the class $3a$ of G and the classes $15e$ and $15f$ of H_7 are fused into the class $15a$ of G .

This forces $\phi|_{H_7} = \chi_{12} + \chi_{13}$.

Finally, let us consider the subgroup $H_{12} = (A_5 \times J_2) : 2$. This group has a unique character χ_1 of degree 1, two characters, χ_8 and χ_{18} , of degree 4, a unique character χ_2 of degree 12, two characters, χ_9 and χ_{10} , of degree 24, while all the other characters have degree greater than 24. Furthermore, this subgroup contains elements belonging to the class $7a$ of G (also labelled $7a$ in H_{12}) for which $\chi_1(7a) = 1$, $\chi_8(7a) = \chi_{18}(7a) = 4$, $\chi_2(7a) = -2$ and $\chi_9(7a) = \chi_{10}(7a) = -4$. So, we have three possible decompositions for $\phi|_{H_{12}}$: $2 \cdot \chi_2$, χ_9 or χ_{10} . Looking at the classes $3a$, $3c$, $5b$, $5c$ and $5d$ of H_{12} , we have a priori the following possibilities:

$\phi _{H_{12}}$	$3a$	$3c$	$5b$	$5c$	$5d$
$2 \cdot \chi_2$	-12	24	-6	24	4
χ_9	-12	-12	-6	-6	-6
χ_{10}	-12	-12	-6	-6	4

However, the classes $3a$ and $3c$ of H_{12} are fused into the class $3a$ of G and the classes $5b$, $5c$ and $5d$ of H_{12} are fused into the class $5a$ of G .

This forces $\phi|_{H_{12}} = \chi_9$.

Now, since every conjugacy class of G has a non-trivial intersection with at least one of the maximal subgroups considered above, we conclude that the value of ϕ is uniquely

determined. In other words, G has a unique irreducible representation of degree 24 in characteristic 2.

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